Dynamical model

INTRODUCTION:

These are prognostic mathematical models, giving time evolution of atmospheric dynamical system, with a known initial condition.

These models can predict the future values of different fields like wind, temperature, humidity, geopotential height, vorticity etc.

For any dynamical model, besides initial conditions, a set of mathematical equations are required.

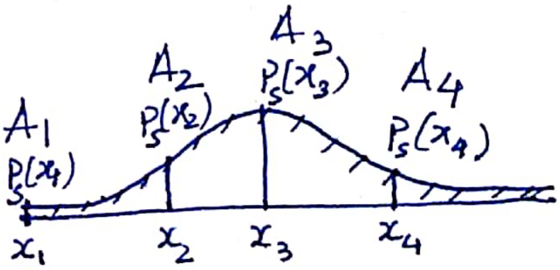
A dynamical model may use all the seven governing equations, viz., equations of motion, thermo-dynamic energy equation, mass continuity equation, moisture continuity equation and the equation of state. Such model is able to give the time evolution of all basic Meteorological parameters, viz., u, v, w, T, ρ, q, p. Such model is known as primitive equation model (PEM).

Besides above, there are a different types of dynamical models, which instead of using basic governing equations, uses other equations, viz., vorticity equation, divergent equation, quasi geostrophic vorticity equation, coupled with diagnostic equation etc. ~~This~~ Such models, can give the time evolution of derived parameters like, vorticity. Such models are known as derived models. For example quasi geostrophic model ~~can prognose~~ consists of two governing equations, viz., quasi geostrophic vorticity equation and diagnostic omega equation. It predicts geostrophic stream function 'ϕ/f_0' and diagnoses 'ω'.

Primitive equation model:

Governing equations in z & p co-ordinates, we have already discussed ~~during~~ in Dynamic Meteorology of Semester-I. There had been an adequate discussion on the merits of using pressure (p) as a vertical co-ordinate.

However, it has a serious limitation in representing lower boundary condition over/near a mountainous region.



To appreciate the limitation, the adjoining figure may be referred to, where a section of a mountain by x-z plane has been shown. ~~Four arbitrary points x_1, x_2, x_3 & x_4 have been shown~~ Four arbitrary points A_1, A_2, A_3 & A_4 ~~are~~ on the mountain surface have been taken with co-ordinates x_1, x_2, x_3 & x_4 along x-axis. If the pressure at these points on the mountain surface are $P_s(x_1)$, $P_s(x_2)$, $P_s(x_3)$ & $P_s(x_4)$ and if 'p' is used as vertical co-ordinate, then as these all four points lie on the lower boundary (the mountain surface), hence it requires,

$P_s(x_1) = P_s(x_2) = P_s(x_3) = P_s(x_4)$. ~~Hower~~ However as these four points are having different heights above mean sea level, hence, hydrostatic conditions doesn't allow so. Thus, even though all these four points lie on the lower boundary surface, but the values of vertical co-ordinate at these points are not same. This is the difficulties in the use of 'p' as a vertical co-ordinate near mountainous region.

To get rid of form above problem, a new vertical co-ordinate is used, viz., Sigma (σ) co-ordinate. At any point (x, y, z) in space, σ is defined by

$$\sigma = \frac{p(x,y,z)}{P_s(x,y)}$$

or

$$\sigma = \frac{P(x,y,z) - P_T(x,y)}{P_s(x,y) - P_T(x,y)}$$

; where P_s is surface pressure and P_T is pressure at the top model level.

~~Using~~ Using above defination, one can easily see that at any arbitrary point (x_i, y_i) on the lower boundary (including mountain surface also) value of

$$\sigma = \frac{P_s(x_i, y_i)}{P_s(x_i, y_i)} = 1$$

and at the top $\sigma = 0$.

Now we shall see how the governing equations look like using 'σ' as a vertical co-ordinate.

1. Hydrostatic approximation:

We know, $\frac{\partial \phi}{\partial p} = -\frac{RT}{p}$

Now $\sigma = \frac{p}{p_s}$

Thus $\frac{\partial ()}{\partial p} \equiv \frac{\partial ()}{\partial \sigma}\frac{\partial \sigma}{\partial p} = \frac{1}{p_s}\frac{\partial ()}{\partial \sigma}$

Thus hydrostatic ~~equation be~~ approximation reduces to,

$$\frac{1}{p_s}\frac{\partial \phi}{\partial \sigma} = -\frac{RT}{\sigma p_s}$$

$$\Rightarrow \boxed{\frac{\partial \phi}{\partial \sigma} = -\frac{RT}{\sigma}} \quad \text{----} (1)$$

2. Mass Continuity equation

In 'p' co-ordinate continuity equation is

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0$$

Now, $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_\sigma$

Now $\omega = \frac{dp}{dt} = \frac{d}{dt}(\sigma p_s) = \dot{\sigma} p_s + \sigma \dot{p}_s$

$$\Rightarrow \frac{\partial \omega}{\partial p} = \frac{1}{p_s}\frac{\partial \omega}{\partial \sigma} = \frac{1}{p_s}\frac{\partial}{\partial \sigma}(\dot{\sigma} p_s + \sigma \dot{p}_s)$$

$= \frac{\partial \dot{\sigma}}{\partial \sigma} + \frac{\dot{p}_s}{p_s}$ [Neither surface pressure (p_s) or it's change ($\dot{p}_s$) depends on σ]

Thus continuity equation reduces to,

$$\boxed{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)_\sigma + \frac{\partial \dot{\sigma}}{\partial \sigma} = -\frac{1}{P_s}\frac{dP_s}{dt}} \quad \text{—— (2)}$$

From the above equation it follows that at surface level, on the wind ward side of a mountain, due to convergence, there is increase of surface pressure. This equation may written as ~~also~~

$$\frac{\partial P_s}{\partial t} + \vec{\nabla}_\sigma \cdot (P_s \vec{V}_H) + P_s \frac{\partial \dot{\sigma}}{\partial \sigma} = 0 \qquad \left[\text{as } \frac{\partial P_s}{\partial \sigma} = 0\right].$$

3. Moisture Continuity equation:

Moisture continuity equation is given by

$$\frac{dq}{dt} = 0 \Rightarrow \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)_p q + \omega\frac{\partial q}{\partial p} = 0$$

Now $\omega \frac{\partial q}{\partial p} = \left(\dot{\sigma} P_s + \sigma \dot{P}_s\right) \cdot \frac{1}{P_s}\frac{\partial q}{\partial \sigma}$

$$= \dot{\sigma}\frac{\partial q}{\partial \sigma} + \frac{\sigma}{P_s}\dot{P}_s \frac{\partial q}{\partial \sigma}$$

$$= \dot{\sigma}\frac{\partial q}{\partial \sigma} + \frac{\partial q}{\partial \ln \sigma}\frac{d(\ln P_s)}{dt}$$

Thus ~~Cont~~ moisture continuity equation reduces to,

$$\frac{d_\sigma q}{dt} + \dot{\sigma}\frac{\partial q}{\partial \sigma} + \frac{\sigma}{P_s}\dot{P}_s\frac{\partial q}{\partial \sigma} = 0$$

$$\Rightarrow \boxed{\frac{d_\sigma q}{dt} + \sigma\frac{\partial q}{\partial \sigma}\left(\frac{\dot{\sigma}}{\sigma} + \frac{\dot{P}_s}{P_s}\right) = 0}$$

Now, $\frac{\partial T}{\partial p} - \frac{\alpha}{c_p} = \frac{\partial T}{\partial p} - \frac{R}{c_p}\frac{T}{p} = \frac{T}{p}\left(\frac{\partial \ln T}{\partial \ln p} - \frac{R}{c_p}\right)$

$$\ln\theta - \ln T = \frac{R}{c_p}\ln 1000 - \frac{R}{c_p}\ln p$$

$$\frac{\partial \ln\theta}{\partial \ln p} - \frac{\partial \ln T}{\partial \ln p} = -\frac{R}{c_p}$$

$$\frac{\partial \ln T}{\partial \ln p} - \frac{R}{c_p} = \frac{\partial(\ln\theta)}{\partial(\ln p)} = \frac{p}{\theta}\frac{\partial\theta}{\partial p}$$

$$\therefore\ \omega\left(\frac{\partial T}{\partial p} - \frac{\alpha}{c_p}\right) = \frac{\omega p}{\theta}\frac{\partial\theta}{\partial p}\cdot\frac{T}{p}$$

$$\cancel{= (\dot\sigma p_s + \sigma\dot p_s)\,\sigma p_s \frac{1}{p_s}\frac{\partial\theta}{\partial\sigma}}$$

$$= \frac{T}{\theta}\frac{\partial\theta}{\partial p}\,\omega$$

$$= \frac{T}{\theta}(\dot\sigma p_s + \sigma\dot p_s)\frac{1}{p_s}\frac{\partial\theta}{\partial\sigma}$$

$$= \dot\sigma\left(\frac{T}{\theta}\frac{\partial\theta}{\partial\sigma}\right) + T\frac{\dot p_s}{p_s}\frac{\partial\ln\theta}{\partial\ln\theta}$$

$$= \left(\dot\sigma + \sigma\frac{\dot p_s}{p_s}\right)S_\sigma$$

where $S_\sigma = \frac{T}{\theta}\frac{\partial\theta}{\partial\sigma} \rightarrow$ static stability in σ-coordinate.

Thus we have

$$\boxed{\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)_\sigma T + \dot\sigma S_\sigma = \frac{\dot Q}{c_p} - \sigma\frac{\dot p_s}{p_s}S_\sigma}$$

Horizontal equation of motion in σ-Coordinate.

We know that horizontal equation of motion in p- coordinate is

$$\left[\frac{\partial \vec{V}_H}{\partial t}+(\vec{V}_H\cdot\vec{\nabla})\vec{V}_H\right]_p+\omega\frac{\partial \vec{V}_H}{\partial p}=-\vec{\nabla}_p\phi-f\hat{k}\times\vec{V}_H+\vec{F}_H$$

where $\vec{V}_H\cdot\vec{\nabla}_p \equiv \left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)_p$

Now, as $\sigma=\frac{p}{p_s}$, hence,

$$\left[\frac{\partial \vec{V}_H}{\partial t}+(\vec{V}_H\cdot\vec{\nabla})\vec{V}_H\right]_p+\omega\frac{\partial \vec{V}_H}{\partial p}=\left[\frac{\partial \vec{V}_H}{\partial t}+(\vec{V}_H\cdot\vec{\nabla})\vec{V}_H\right]_\sigma+\dot{\sigma}\frac{\partial \vec{V}_H}{\partial \sigma}$$

Now, we know that for a general vertical co-ordinate, say, 'r'; and for an arbitrary function $A(x,y,z(x,y,r,t),t)$ we have,

$$\left[\frac{\partial A}{\partial s}\right]_r=\left[\frac{\partial A}{\partial s}\right]_z+\frac{\partial A}{\partial r}\frac{\partial r}{\partial z}\left[\frac{\partial z}{\partial s}\right]_r.$$

Let us substitute, $r=\sigma$ & $A=p$ and $s=x, y$ & t.

~~So, we~~

So, we have,

$$\left(\frac{\partial p}{\partial x}\right)_z=\left(\frac{\partial p}{\partial x}\right)_\sigma \cancel{+} - \frac{\partial p}{\partial \sigma}\frac{\partial \sigma}{\partial z}\left(\frac{\partial z}{\partial x}\right)_\sigma$$

$$=\cancel{\phi}\ \sigma\left(\frac{\partial p_s}{\partial x}\right)_\sigma+p_s\cdot\frac{1}{p_s}(-gp)\left(\frac{\partial z}{\partial x}\right)_\sigma$$

$$\Rightarrow \frac{1}{p}\left(\frac{\partial p}{\partial x}\right)_z=\frac{pRT}{p_s p}\left(\frac{\partial p_s}{\partial x}\right)_\sigma \cancel{\phi} - \left(\frac{\partial \phi}{\partial x}\right)_\sigma$$

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = \frac{RT}{P_s}\left(\frac{\partial P_s}{\partial x}\right)_\sigma - \left(\frac{\partial \phi}{\partial x}\right)_\sigma$$

$$\Rightarrow \cancel{\frac{1}{\rho}\vec{\nabla}_p z = \vec{\nabla}}$$

$$\Rightarrow \frac{1}{\rho}\vec{\nabla}_z p = \vec{\nabla}_p \phi = RT\,\vec{\nabla}_\sigma \ln P_s - \vec{\nabla}_\sigma \phi .$$

Hence, horizontal equation of motion in σ- coordinate is given by,

$$\boxed{\left[\frac{\partial \vec{V}_H}{\partial t} + \left(\vec{V}_H \cdot \vec{\nabla}\right)\vec{V}_H\right]_\sigma + \dot{\sigma}\frac{\partial \vec{V}_H}{\partial \sigma} = RT\,\vec{\nabla}_\sigma \ln P_s - \vec{\nabla}_\sigma \phi - f\hat{k}\times\vec{V}_H + \vec{F}_s}$$

Derived model

Quasigeostrophic model, we have already discussed in the discussion of GFD.

Here we shall discuss Barotropic model, Equivalent Barotropic model and 2-layer Baroclinic model.

- **Barotropic model**

Governing equation.

Non-divergent / large scale Barotropic vorticity equation:

$$\frac{\partial \zeta}{\partial t} + \vec{V}_H \cdot \vec{\nabla}_p(\zeta + f) = 0$$

As we consider non-divergent flow, hence

$$\vec{V}_H = \vec{V}_\psi = \hat{k} \times \vec{\nabla}\psi.$$

So, $\zeta = \hat{k} \cdot \vec{\nabla} \times \vec{V}_H = \nabla^2 \psi$

Hence above equation is reduced to

$$\left(\frac{\partial}{\partial t} + \vec{V}_\psi \cdot \vec{\nabla}_p\right)(\nabla^2 \psi + f) = 0$$

Now, $(\vec{V}_\psi \cdot \vec{\nabla}_p)(\nabla^2 \psi + f)$

$$= [(\hat{k} \times \vec{\nabla}\psi) \cdot \vec{\nabla}_p](\nabla^2 \psi + f)$$

$$= -\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2 \psi + f) + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2 \psi + f)$$

$$= \begin{vmatrix} \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} \\ \frac{\partial}{\partial x}(\nabla^2 \psi + f) & \frac{\partial}{\partial y}(\nabla^2 \psi + f) \end{vmatrix} = J[\psi, (\nabla^2 \psi + f)].$$

Thus the governing equation reduces to,

$$\boxed{\frac{\partial}{\partial t}(\nabla^2\psi) = -J(\psi, \nabla^2\psi + f)} \rightarrow \text{Model equation.}$$

$$\Downarrow$$

$$\boxed{\nabla^2\left(\frac{\partial\psi}{\partial t}\right) = -J(\psi, \nabla^2\psi + f)}$$

How to predict ψ at a point?

Step I : observation of geopotential height ($\phi = gz$)

Step II : Prepare balanced gridded value of ϕ.

Step III : Computation of initial values $\psi = \frac{\phi}{f_0}$ at each point.

Step IV : Computation of $\cancel{\nabla^2(\psi+f)}$ $(\nabla^2\psi + f)$ at each point at initial time, say $t = t_0$.

Step V : Computation of $J(\psi, \nabla^2\psi + f)$ using Arakawa Jacobian.

Step VI : Solve above Poisson's equation numerically, using Relaxation method, at each grid point (i,j), for $\left(\frac{\partial\psi}{\partial t}\right)$.

Step VII ; If $\left(\frac{\partial\psi}{\partial t}\right)^0_{(i,j)} = C^0_{ij}$, then

we have,
$$\frac{\psi^{0+1}_{(i,j)} - \psi^0_{(i,j)}}{\Delta t} = C^0_{ij}$$

$$\Rightarrow \psi'_{(i,j)} = \psi^0_{(i,j)} + C^0_{ij}\,\Delta t$$

As, $\psi^{0}_{(i,j)}$, c^{0}_{ij} and Δt all are known, hence (11)

$\psi^{1}_{(i,j)}$ Can be found out.

Similarly, knowing $\psi^{n}_{(i,j)}$, we can find

$$\psi^{n+1}_{(i,j)} = \psi^{n}_{(i,j)} + c^{(n)}_{ij}\,\Delta t$$

where $\cancel{c^{(n)}_{ij}}$ $c^{n}_{ij} = \left(\frac{\partial \psi}{\partial t}\right)^{n}_{(i,j)}$. ~~obtained~~

By knowing ψ at each grid point for future time, geopotential height ~~of~~ at each grid point ~~at ch~~ at the level of non-divergence Can be predicted.

Note: This is the first successful operational NWP model developed by a group of Scientists in Princeton university, USA, viz. J. Charney, Von-Neumann & FJØrtoft, in 1940's.

However, this model has a serious limitation that entire atmosphere is represented by the LND only. Also, barotropic condition doesn't allow temperature advection, a serious limitation.

Equivalent Barotropic model

To overcome, at least partially, the limitations in the above discussed Barotropic model, in 1949-50, they proposed this model, by partially incorporating the ~~effect~~ Baroclinic condition, as follows:

Let the horizontal wind at an arbitrary point (x, y, p) at any time 't' be expressed as

$$\vec{V}(x, y, p, t) = A(p)\,\vec{V}_M(x, y, t) \quad \text{--- (1)}$$

where, $$\vec{V}_M(x, y, t) = \frac{1}{P_s}\int_0^{P_s} \vec{V}(x, y, p, t)\,dp \quad \text{--- (2)}$$

~~So, that~~ Thus $$\frac{1}{P_s}\int_0^{P_s} A(p)\,dp = 1 \quad \text{--- (3)}$$

$$\therefore \zeta = \hat{k}\times\vec{V} = A(p)\,\hat{k}\times\vec{V}_M \qquad \text{[Prove it].}$$
$$= A\,\zeta_M$$

Now, after scale analysis, vorticity equation for large scale motion is

$$\frac{\partial \zeta}{\partial t} + \vec{V}_H\cdot\vec{\nabla}(\zeta+f) = f_0\frac{\partial \omega}{\partial p}$$

$$\Rightarrow A\frac{\partial \zeta_M}{\partial t} + A^2\vec{V}_M\cdot\vec{\nabla}\zeta_M + A\vec{V}_M\cdot\vec{\nabla}f = f_0\frac{\partial \omega}{\partial p}$$

$$\Rightarrow \frac{\partial \zeta_M}{\partial t}\frac{1}{P_s}\int_0^{P_s} A\,dp + \left(\vec{V}_M\cdot\vec{\nabla}\zeta_M\right)\frac{1}{P_s}\int_0^{P_s} A^2\,dp + \left(\vec{V}_M\cdot\vec{\nabla}f\right)\frac{1}{P_s}\int_0^{P_s} A\,dp$$
$$= \frac{f_0}{P_s}\int_0^{P_s}\frac{\partial \omega}{\partial p}\,dp$$

$$\Rightarrow \frac{\partial \zeta_M}{\partial t} + \overline{A^2}\, \vec{V}_M \cdot \vec{\nabla} \zeta_M + \vec{V}_M \cdot \vec{\nabla} f = \frac{f_0}{P_s} \omega(P_s). \quad \text{---} (4)$$

where $\overline{A^2} = \frac{1}{P_s} \int_0^{P_s} [A(P)]^2 \, dp$

multiplying both sides of above equation by $\overline{A^2}$ (= A^*, say) and using the fact that for large scale motion, $\omega(P_s) \approx -g \rho_s w(z_s)$

we get

$$\Rightarrow A^* \frac{\partial \zeta_M}{\partial t} + A^* \vec{V}_M \cdot \vec{\nabla}(A^* \zeta_M) + A^* \vec{V}_M \cdot \vec{\nabla} f = -\frac{g A^* \rho_s w(z_s)}{P_s} \quad \text{--} (5)$$

Now $w(z_s) \approx \frac{\partial z_s}{\partial t} + \vec{V}_H \cdot \vec{\nabla} z_s$

$\approx \vec{V}_H \cdot \vec{\nabla} z_s$. This is also very small, except in the mountainous area. More over ~~denomitor~~ denominator of RHS is very ~~small~~ large, as compared to the numerator, hence may be ignored for large scale motion.

Hence, (5) reduces to,

$$\frac{\partial}{\partial t}(A^* \zeta_M) + (A^* \vec{V}_M) \cdot \vec{\nabla}(A^* \zeta_M + f) \approx 0 \quad \text{---} (6)$$

Now, let us define $(\ )^* = A^* (\ )_M$.

Then equation (6) reduces to,

$$\frac{\partial \zeta^*}{\partial t} + \vec{V}^* \cdot \vec{\nabla}(\zeta^* + f) \simeq 0 \quad \text{--- (7)}$$

~~Equation (7)~~

This is the governing equation for equivalent Barotropic model.

~~This~~ The equation looks similer to the governing equation for non-divergent barotropic model:

$$\frac{\partial \zeta}{\partial t} + \vec{V}_H \cdot \vec{\nabla}(\zeta + f) = 0 \quad \text{--- (8)}$$

Thus applying equation (7) is equivalent to apply (8) at a level, where,

~~to~~

$$(\ )^* = (\ )$$

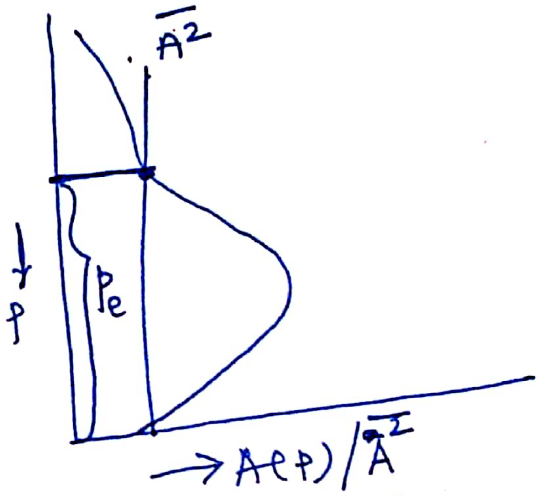
$$\Rightarrow A^* (\ )_M = A(p) (\ )_M$$

$$\Rightarrow A^* = A(p). \quad \text{--- (9)}$$

The level, say $p = p_e$, where $A^* = A(p_e)$, is called equivalent level.

Thus application of equivalent Barotropic model is equivalent to application of Barotropic model at equivalent level, instead of applying at level of non divergence.

Thus from the different level of horizontal wind data, one has to find $A(p)$ for each level. Then $A(p)$ should be plotted against 'p' and a curve will be obtained. Then, on it plot $\bar{A}^2 = \text{Const}$. Where these two intersect, that is equivalent level.



~~Them~~ Then algorithm is same as Barotropic model.

Thus using this model geopotential height of equivalent level can be predicted. Equivalent level is approximately 500 hpa, slightly above level of non-divergence.

Baroclinic model

Already discussed in the content of Baroclinic instability.

Assignments: Write algorithms for prediction of geopotential heights for 250 hpa, 750 hpa.